288. [Prof. J. H. Kershner, E. J. Edmunds, Geo. Eastwood and Dr. H. Eggers, each sent a solution of this question, but they all lead to complicated equations of a high degree, and are too long for the space that remains at our disposal in this No. We give below the result obtained by Mr. Eastwood.]

Putting r and R for the radii of the two circles, whose centers are O_1 and O_2, O_1 being the origin and (x_1, y_1). (x_2, y_2) coordinates of M and N respectively; also putting $AC=a$ and $AB=\beta$, the equation of line MN is

$$y-y_1=\frac{y_1-y_2}{x_1-x_2}(x-x_1). \tag{7}$$

Substituting for x_1, y_1, x_2, y_2 from equations (1), (2), (5) and (6) we get

$$[y(a^2+r^2)-r(a^2-r^2)].[(2R^2r^2a-\beta a^2-\beta r^2)(\beta-a)^2+R^2(\beta-2a)a^2+\beta R^2r^2]$$
$$-[2r(R^2+\beta-a)a^2-2R(a^2+r^2)(\beta-a)^2].[x(a^2+r^2)-2R^2a]=0. \tag{8}$$

The differential coefficient of (8), with respect to a, equated to zero gives an equation of the 5th degree in a, any real root of which substituted in (8) will give the envelop of MN.

If P denote the middle point of MN the equation of the line CP will be

$$y-r=\frac{r-\frac{1}{2}(y_1-y_2)}{a-\frac{1}{2}(x_1+x_2)}(x-a).$$

Combining this equation with (7) we get for the eqn. of the locus of P

$$2y-(r+y_1)=\frac{y_1-y_2}{x_1-x_2}(x-x_1)+\frac{r-\frac{1}{2}(y_1+y_2)}{a-\frac{1}{2}(x_1+x_2)}(x-a). \tag{10}$$

SOLUTIONS OF PROBLEMS IN NUMBER ONE.

SOLUTIONS of problems in No. 1 have been received as follows:

From R. J. Adcock, 289, 292, 295; Prof. R. C. Arendt, 295; Prof. W. W. Beman, 295; Marcus Baker, 289; Alex. S. Christie, 291; Dr. H. Eggers, 295; Prof. A. B. Evens, 295; George Eastwood, 295; Prof. A. Hall, 295; Prof. W. W. Johnson, 289, 294, 295; Prof. J. H. Kershner, 289, 291, 295; Chas. H. Kummell, 289, 291, 292, 295; Prof. H. T. J. Ludwick, 295; Prof. J. Scheffer, 289, 295; Prof. E. B. Seitz, 289, 291, 295; Walter Siverly, 289; Geo. Lilley, 289; Prof. W. P. Casey, 289, 295.

SOLUTION BY GEORGE LILLEY, A. M., CORNING, IOWA.

Multiplying together the two well known formulæ of spherical trigonometry,

$$\left.\begin{aligned}\cos a &= \cos b\cos c+\sin b\sin c\cos A\\ \sin a\cos B &= \sin c\cos b-\cos c\sin b\cos A\end{aligned}\right\}$$, we readily obtain

$$\tan c=\frac{\cot a\cos A+\cot b\cos B}{\cot a\cot b-\cos A\cos B}.$$

Hence the proposition is incorrectly stated.

[All our correspondents, who are credited with solutions of this problem, have obtained the same value for tan c as is here given by Mr. Lilley.]

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290. No solution received.

291. "If ABC, $A'B'C'$ are two trirectangular triangles on the surface of a sphere (the letters being arranged in the same order of rotation); show that $\cos AA' = \cos BB'\cos CC' - \cos B'C\cos BC'$."

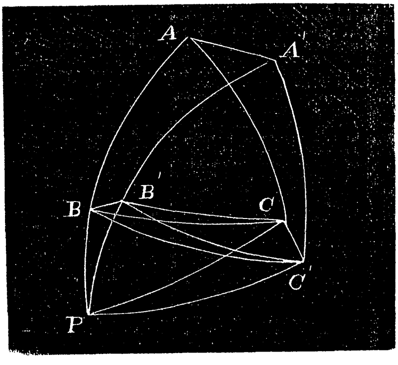
SOLUTION BY ALEX. S. CHRISTIE, U. S. COAST SURVEY, WASH., D. C.

Adopting the usual quaternion notation, and putting in all the steps of the process:

$$\begin{aligned}\cos AA' &= -S\alpha\alpha' = -S.V\beta\gamma V\beta'\gamma'\\ &= S\gamma V\beta V\beta'\gamma' = S\gamma(\gamma' S\beta\beta' - \beta' S\beta\gamma')\\ &= S\beta\beta' S\gamma\gamma' - S\beta'\gamma S\beta\gamma'\\ &= \cos BB'\cos CC' - \cos B'C\cos BC'\end{aligned}$$

when $\alpha = V\beta\gamma$ and $\alpha' = V\beta'\gamma'$, or the triangles lettered in the *same* order of rotation; but $= \cos BC'\cos B'C - \cos BB'\cos CC'$ when $\alpha = V\beta\gamma$ and $\alpha' = V\gamma'\beta'$, or the triangles lettered in the *opposite* order.

SOLUTION BY PROF. E. B. SEITZ, KIRKSVILLE, MO.



Produce AB and $A'B'$ to P, and draw the arcs PC and PC', each of which is a quadrant.

Since BPC and $B'PC'$ are right angles, the angles BPB' and CPC' are equal; hence

$$\cos BPB' = \cos CPC' = \cos CC',$$

$$\cos B'PC = \sin CPC' = \sin CC',$$

and $$\cos BPC' = -\sin CPC' = -\sin CC'$$

In the triangles PBB' and PAA' we have

$$\cos BB' = \cos PB \cos PB' + \sin PB \sin PB' \cos CC', \qquad (1)$$

$$\cos AA' = \cos PA \cos PA' + \sin PA \sin PA' \cos CC'$$

$$= \sin PB \sin PB' + \cos PB \cos PB' \cos CC'. \qquad (2)$$

In the triangles $PB'C$ and PBC' we have

$$\cos B'C = \sin PB' \cos B'PC = \sin PB' \sin CC' \qquad (3)$$

$$\cos BC' = \sin PB \cos BPC' = -\sin PB \sin CC'. \qquad (4)$$

Substituting the values of $\sin PB'$ and $\sin PB$ from (3) and (4) in (1) and (2), and reducing, we find

$$\cos BB' \sin^2 CC' = \cos PB \cos PB' \sin^2 CC' - \cos B'C \cos BC' \cos CC', \qquad (5)$$

$$\cos AA' \sin^2 CC' = \cos PB \cos PB' \sin^2 CC' \cos CC' - \cos B'C \cos BC'. \qquad (6)$$

Subtracting (5) multiplied by $\cos CC'$ from (6), and dividing by $\sin^2 CC'$, we have

$$\cos AA' = \cos BB' \cos CC' - \cos B'C \cos BC'.$$

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292. "Two surveyors measure a plane quadrangular field, one measuring the four sides a, b, c, d with a chain and the other, the angles (ab), (bc), (cd), (da) with a theodolite. From former experience it is known that the first is liable to a probable error of m inches per chain and the other to a probable error of n'' per angle. Required the weights of the linear and angular measurements, also conditions to be fulfilled by the measured quantities in approximate linear form and the analytical formation of the normal equations for determining the most probable corrections to the measured quantities."

SOLUTION BY R. J. ADCOCK, ROSEVILLE, ILL.

Let the true or required values of the measures a, b, c, d, be x, y, z, u, and the measured angles (ab), (bc), (cd), (da), denote by A_1, B_1, C_1, D_1, and the required values of these angles by A, B, C, D. Since the angles are separate independent measures and have a condition independent of the sides, they are to be considered separately. Then, by the principle of Least Squares announced at page 22,

$$(A-A_1)^2+(B-B_1)^2+(C-C_1)^2+(D-D_1)^2 = \text{a minimum}. \qquad (1)$$

Eliminating D by the equation

$$A+B+C+D = 2\pi, \qquad (2)$$

and differentiating with respect to each of the other variables A, B, C, and adding the resulting equations, gives

$$A+B+C = \tfrac{3}{2}\pi + A_1 + B_1 + C_1.$$

Hence by (2) $D=\frac{1}{2}\pi+\frac{3}{4}D_1-\frac{1}{4}(A_1+B_1+C_1)$, $C=\frac{1}{2}\pi+\frac{3}{4}C_1-\frac{1}{4}(A_1+B_1+D_1)$, $B=\frac{1}{2}\pi+\frac{3}{4}B_1-\frac{1}{4}(A_1+C_1+D_1)$, $A=\frac{1}{2}\pi+\frac{3}{4}A_1-\frac{1}{4}(B_1+C_1+D_1)$.

For the sides, by the principle above alluded to,

$$(x-a)^2+(y-b)^2+(z-c)^2+(u-d)^2=\text{a minimum}; \qquad (3)$$

from which two variables may be eliminated by the trigonometrical relations

$$z=\frac{u\sin D-y\sin A}{\sin(C+D)}=nu-ry,\ x=\frac{u\sin C-y\sin D}{\sin(C+D)}=n_1u-r_1y, \qquad (4)$$

giving $(n_1u-r_1y-a)^2+(y-b)^2+(nu-ry-c)^2+(u-d)^2=$ a minimum.

Differentiating and reducing,

$$y=\frac{(n_1a+nc+d)(nr+n_1r_1)-(r_1a+rc-b)(n^2+n_1^2+1)}{(r^2+r_1^2)(n^2+n_1^2+1)-(nr+n_1r_1)^2}.$$

$$2\,\text{area}=ab\sin A_1+cd\sin C_1=bc\sin B_1+ad\sin D_1$$
$$=\tfrac{1}{2}(ab\sin A_1+bc\sin B_1+cd\sin C_1+ad\sin D_1),\ \text{nearly.}$$

Differentiating, and using the probable errors m inches and n'' for the differentials, gives

$$\frac{n''}{m}=2\,\frac{bc\sin B_1+ad\sin D_1-ab\sin A_1-cd\sin C_1}{ab\cos A_1+cd\cos C_1-bc\cos B_1-ad\cos D_1},$$

the square of which gives the relative weights required.

[Mr. Kummell has sent us an extended general solution of this problem which is omitted here for want of room but will be inserted in a future No. He also sent a solution of 282 with condition (3) modified, which will appear in some future number.]

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293. No solution received.

294. "One curve rolls upon another; prove that a series of carried parallel curves envelope a series of parallel curves, or, involutes of the same evolute envelope involutes of the same evolute."

SOLUTION BY. PROF. JOHNSON.

Parallel curves, or involutes of a common evolute, have common normals, and intercept upon these normals a constant length. Consider the normal (or normals) of the carried curve which at any instant pass through the instantaneous centre (point of contact of rolling curves).

Since every carried point is moving in a direction perpendicular to the line which joins it with the instantaneous centre, the points where the normal cuts the parallel carried curves are moving in the directions of the tangents to these curves, and will therefore be points of contact with their envelopes. These envelopes have therefor common normals, and are parallel curves.

295. "Given the common astronomical equations

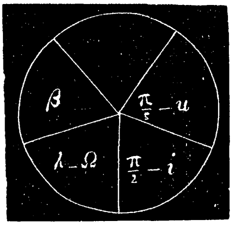
$$\tan g\,(\lambda - \Omega) = \cos i \tan g\, u,$$
$$\sin \beta = \sin i \quad \sin u,$$

eliminate u, and show in this manner that

$$\tan g\, \beta = \tan g\ \ i \sin (\lambda - \Omega)."$$

SOLUTION BY CHAS. H. KUMMELL, U. S. LAKE SURVEY, DETROIT, MICH.

1. The first two relations are consistent with Napier's rules of circular parts as exhibited in the figure. The third is then true by the same rules.



2. $$\tan \beta = \frac{\sin \beta}{\sqrt{(1 - \sin^2 \beta)}} = \frac{\sin i \sin u}{\sqrt{(1 - \sin^2 i \sin^2 u)}}$$
$$= \frac{\sin i \tan u}{\sqrt{(1 + \cos^2 i \tan^2 u)}} = \frac{\tan i \tan(\lambda - \Omega)}{\sqrt{(1 + \tan^2(\lambda - \Omega)}}$$
$$= \tan i \sin (\lambda - \Omega).$$

SOLUTION BY PROF. A. B. EVENS, LOCKPORT, N. Y.

For the sake of brevity put x for $(\lambda - \Omega)$. The first and second equations give, respectively,

$$\tan u = \frac{\tan x}{\cos i}, \quad \sin^2 \beta = \frac{\tan^2 \beta}{1 + \tan^2 \beta} = \frac{\sin^2 i \tan^2 u}{1 + \tan^2 u}.$$

Eliminating $\tan^2 u$ and solving for $\tan^2 \beta$, we find

$$\tan^2 \beta = \frac{\tan^2 i \tan^2 x}{1 + \tan^2 x} = \tan^2 i \sin^2 x;$$

and therefore, restoring the value of x

$$\tan \beta = \tan g\, i \sin (\lambda - \Omega).$$

SOLUTION BY PROF. ASAPH HALL.

From the first equation

$$\frac{\sin^2(\lambda - \Omega)}{\cos^2(\lambda - \Omega)} = \frac{\cos^2 i \sin^2 u}{\cos^2 u}.$$

Adding unity to each side and putting $D = \sqrt{(\cos^2 u + \cos^2 i \sin^2 u)}$, we find

$$\sin (\lambda - \Omega) = \frac{\cos i \sin u}{D}: \qquad \cos (\lambda - \Omega) = \frac{\cos u}{D}.$$

From the second equation, $1 - \sin^2 \beta = \cos^2 \beta = \cos^2 u + \cos^2 i \sin^2 u$ or we have

$$\cos \beta = D.$$

$$\therefore \frac{\sin (\lambda - \Omega)}{\sin \beta} = \frac{\cos i}{\cos \beta \sin i},$$

or $$\tan g\, \beta = \tan g\, i \sin (\lambda - \Omega).$$

[Solutions equally brief and elegant were sent by Professors Beman, Seitz and Johnson.]